is valid for any real value of n. Since in the above considerations no restrictions were put upon x, (F) holds for any value of x, real or complex, if we assume, as we may, that (f) holds for such values of the variables. Thus we may write:

$$f(z) = f(u + vi) = f(u) + f(vi) + 2cuvi. \tag{8}$$

But

$$f(u) = f(u \cdot 1) = uf(1) + u(u - 1)c(1)^2,$$

$$f(vi) = f(v \cdot i) = vf(i) + v(v - 1)c\,(i)^2;$$

hence, (8) may be written:

$$f(z) = f(u + vi) = uf(1) + vf(i) + c(u + vi)^2 - (u - v)c. \tag{9}$$

In order to determine the value of $f(1)$ we put $u = 1$, and $v = 0$ and we have from (9) $f(1) = f(1)$, which shows that the value of $f(1)$ is arbitrary. By putting $u = 0$, $v = 1$,we may show in the same way that $f(i)$ is arbitrary. Hence denoting $f(1) - c$ by α and $f(i) + c$ by β, where α and β are two arbitrary quantities, (9) may be written

$$f(u + vi) = c(u + vi)^2 + \alpha u + \beta v.$$

If $\alpha = \beta = 0$, *i. e.*, if $f(1) = 1$ and $f(i) = -1$, and $c = 1$ we have

$$f(u + vi) = (u + vi)^2.$$

Also variously solved by T. M. SIMPSON, O. S. ADAMS, ELIJAH SWIFT, E. R. SMITH, HORACE OLSON, A. A. BENNETT, C. F. GUMMER, and J. L. WALSH.

468. Proposed by H. C. FEEMSTER, York College, Neb.

In each of the following series find the nth term and sum:

$$2 + 5 + 9 + 15 + 24 + \cdots, \tag{a}$$

$$1 + 6 + 10 + 20 + 35 + \cdots, \tag{b}$$

$$1 + 5 + 15 + 35 + 70 + \cdots, \tag{c}$$

SOLUTION BY J. L. RILEY, Tahlequah, Okla.

(*a*) Using the method of differences we have

$$\begin{array}{ccccccc} 3 & & 4 & & 6 & & 9\cdots \\ & 1 & & 2 & & 3 & \cdot\cdots \\ & & 1 & & 1 & & \cdot\cdot\cdots \\ & & & 0 & & & \cdot\cdot\cdot\cdots \end{array}$$

$$U_n = 2 + 3(n - 1) + \frac{(n - 1)(n - 2)}{\underline{|2}} + \frac{(n - 1)(n - 2)(n - 3)}{\underline{|3}}$$

$$= \frac{n^3 - 3n^2 + 20n - 6}{6}, \text{ the } n\text{th term,}$$

$$S_n = 2n + \frac{3n(n - 1)}{\underline{|2}} + \frac{n(n - 1)(n - 2)}{\underline{|3}} + \frac{n(n - 1)(n - 2)(n - 3)}{\underline{|4}}$$

$$= \frac{n}{24}(n^3 - 2n^2 + 35n + 14), \text{ the sum.}$$

(*b*) In the series $1 + 6 + 10 + 20 + 35 + \cdots$, let $U_n = A + Bn + Cn^2 + Dn^3 + En^4$. Then

$$\begin{cases} A + B + C + D + E = 1, \\ A + 2B + 4C + 8D + 16E = 6, \\ A + 3B + 9C + 27D + 81E = 10, \\ A + 4B + 16C + 64D + 256E = 20, \\ A + 5B + 25C + 125D + 625E = 35. \end{cases}$$

From these equations we find the values of A, B, C, D, E; whence,

$$U_n = -20 + 36n - \frac{115n^2}{6} + \frac{9n^3}{2} - \frac{n^4}{3}.$$

From this we get the 6th term to be 46.

Thus, the series is

$$1 + 6 + 10 + 20 + 35 + 46 + \cdots.$$

Using the method of differences we get

$$\begin{array}{cccccc} 5 & 4 & 10 & 15 & 11 & \cdots \\ & -1 & 6 & 5 & -4 & \cdot\ \cdots \\ & & 7 & -1 & -9 & \cdot\ \cdot\ \cdots. \\ & & & -8 & -8 & \cdot\ \cdot\ \cdot\ \cdots \\ & & & & 0 & \cdot\ \cdot\ \cdot\ \cdot\ \cdots \end{array}$$

Then

$$S_n = n + \frac{5n(n-1)}{\lfloor 2} - \frac{n(n-1)(n-2)}{\lfloor 3} + \frac{7n(n-1)(n-2)(n-3)}{\lfloor 4} - \frac{8n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}.$$

$$= -\frac{n}{120}(8n^4 - 115n^3 + 510n^2 - 1145n + 622).$$

(c) In the series $1 + 5 + 15 + 35 + 70 + \cdots$

$$U_n = A + Bn + Cn^2 + Dn^3 + En^4 + \cdots.$$

$$\left\{\begin{aligned} &A + B + C + D + E = 1, \\ &A + 2B + 4C + 8D + 16E = 5, \\ &A + 3B + 9C + 27D + 81E = 15, \\ &A + 4B + 16C + 64D + 256E = 35, \\ &A + 5B + 25C + 125D + 625E = 70, \end{aligned}\right\} \begin{aligned} A &= 0, \\ B &= 1/4, \\ C &= 11/24, \\ D &= 1/4, \\ E &= 1/24. \end{aligned}$$

Hence,

$$U_n = \frac{n}{4} + \frac{11n^2}{24} + \frac{n^3}{4} + \frac{n^4}{24}.$$

Then

$$24S_n = 6\Sigma n + 11\Sigma n^2 + 6\Sigma n^3 + \Sigma n^4,$$

$$24S_n = 3n(n+1) + \frac{11n(n+1)(2n+1)}{6} + \frac{3n^2(n+1)^2}{2} + \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}.$$

$$S_n = \frac{n}{120}(n^4 + 10n^3 + 35n^2 + 50n + 24) = \frac{n}{120}(n+1)(n+2)(n+3)(n+4).$$

Also solved by O. S. ADAMS, H. H. CONWELL, PAUL CAPRON, and WILLIAM TIER.

GEOMETRY.

496. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find all the lines such that the pairs of tangent planes to a given sphere (ellipsoid) passing through them, shall be orthogonal.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

One obvious solution is given when one of the tangent planes is fixed in position, for then this plane is the locus of lines common to it and a second tangent plane orthogonal to the first.

It is not troublesome to show that the equation of a first plane embracing a line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \tag{1}$$

$$\frac{(x-a)\lambda_1}{l} + \frac{(y-b)\mu_1}{m} + \frac{(z-c)\nu_1}{n} = 0, \tag{2}$$